Tutorial 7
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BACK

Questions of today

1. We recall two formula for the Gamma fuction. The first formula is Exercise 1 on page 174 of the text book:

$$\Gamma(z) = \lim_{z\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

A proof of this can be found in Tutorial 5. An important consequence of the formula is

$$\Gamma(\bar{z}) = \overline{\Gamma(z)}.$$

The other formula is

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

which is theorem 1.4 of Lecture 10.
Show, for $b \in \mathbb{R}$,

$$|\Gamma(bi)|^2 = \frac{\pi}{b\sinh(b\pi)}$$

2. Show that

$$\log\zeta(s) = s\int_2^\infty \frac{\pi(x)}{x(x^s-1)}dx.$$

3. Consider the functional equation

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s).$$

A direct consequence of the functional equation is that $\zeta(-2n) = 0$ for $s \in \mathbb{Z}_{>0}$. Recall

$$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$$

is a meromorphic function on $\mathbb{C}$ with simple poles at $s=0$ and $s=1$. To work with entire functions, let us define $\tilde{\xi}(s) = s(1-s)\xi(s)$, and $\Xi(s) = \tilde{\xi}(1/2+is)$.

   a. Show that the functional equation is equivalent to the statement that $\Xi(s)$ is an even function.
   b. In the midterm question 3, you showed that $\xi(s)$ is of growth order 1, and thus $\Xi(s)$ is also of growth order 1. Deduce that $\zeta(s)$ has infinitely many zeros in the strip $0 \le \mathrm{Re}(s) \le 1$.

4. Recall the formula (HW 3), for $\mathrm{Re}(s) \ge 1$,

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}dx$$

Consider the integral of

$$\frac{z^{s-1}}{e^z-1}$$

over the counter C which consists of three parts, the first part is the part of the real axis from ∞ to some small positive δ, the second part is the circle $|z| = \delta$ in anticlockwise direction, the last part is the part of real axis from δ to ∞. Show that

$$\zeta(s) = \frac{e^{-\pi is}\Gamma(1-s)}{2\pi i}\int_C \frac{z^{s-1}}{e^z-1}dz.$$

5. Let us write the Taylor expansion of $z/(e^z-1)$ as

$$\frac{z}{e^z-1} = 1 + B_1 z + B_2\frac{z^2}{2!} + B_3\frac{z^3}{3!} + \cdots$$

Using the previous question, show the following

   a. $\zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1}$ for $n \in \mathbb{Z}_{\ge 0}$.
   b. $\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$ for $n \in \mathbb{Z}_{>0}$.
   c. $\zeta'(0) = -\frac{1}{2}\log 2\pi$.

6. Using question 3 to show that

$$\zeta(s) = \frac{Ae^{bs}}{(s-1)\Gamma(1+s/2)}\prod\left(1-\frac{s}{\rho}\right)e^{s\rho},$$

for some constant A, b. Show that $A = \frac{1}{2}$, and $b = \log 2\pi - 1 - \frac{1}{2}\gamma$.

Hints & solutions of today

1. Note that $|\Gamma(z)|^2 = \Gamma(z)\Gamma(\bar{(z)})$ by the first formula. Now, using the second formula

$$\begin{aligned}\Gamma(1-bi)\Gamma(bi) &= \frac{\pi}{\sin(b\pi i)}\\ -bi\Gamma(-bi)\Gamma(bi) &= \frac{\pi}{\sin(b\pi i)}\\ \Gamma(-bi)\Gamma(bi) &= \frac{\pi}{-bi\sin(b\pi i)} = \frac{\pi}{b\sinh(b\pi)}\end{aligned}$$

2.

$$\begin{aligned}\log\zeta(s) &= \log\left(\prod_p \frac{1}{1-\frac{1}{p^s}}\right)\\ &= -\sum_p \log\left(1-\frac{1}{p^s}\right)\\ &= -\sum_{n=2}^\infty(\pi(n)-\pi(n-1))\log\left(1-\frac{1}{n^s}\right)\\ &= -\sum_{n=2}^\infty \pi(n)\left[\log\left(1-\frac{1}{n^s}\right) - \log\left(1-\frac{1}{(n+1)^s}\right)\right]\\ &= \sum_{n=2}^\infty \pi(n)\int_n^{n+1}\frac{s}{x(x^s-1)}dx\\ &= s\int_2^\infty \frac{\pi(x)}{x(x^s-1)}dx\end{aligned}$$

3. a. By a direct substitution.
   b. First argue $\xi(s)$ has infinitely many zeros as follow: Since $\Xi(s)$ is an even function, $\Xi(\sqrt{s})$ is well-defined. Then note that $\Xi(\sqrt{s})$ has order $\frac{1}{2}$, and so has infinitely many zeros by Homework 2 question 5.
   Next argue that $\xi(s)$ has no zeros outside the range $0 \le \mathrm{Re}(s) \le 1$ using the formula

$$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$$

   Finally, use the above formula again to conclude.

4. Let $I(s) = \int_C \frac{z^{s-1}}{e^z-1}dz$. An application of Cauchy's theorem would tell that $I(s)$ is independent of δ, so we may take $\delta \to 0$. From $\mathbb{Re}(s) \ge 1$, we can also see that the part of the integral over the circle is $\to 0$ when $\delta \to 0$. Therefore,

$$I(s) = -\int_0^\infty \frac{x^{s-1}}{e^z-1}dx + + \int_0^\infty \frac{(xe^{2\pi i})^{s-1}}{e^z-1}dx.$$

(Note the negative sign of the first integral is due to its orientation, also note that we need an extra $e^{2\pi i}$ for the change of argument of $\log$ in the second integral.) And hence

$$I(s) = (e^{2\pi is}-1)\zeta(s)\Gamma(s)$$

Therefore,

$$\begin{aligned}\zeta(s) &= \frac{1}{(e^{2\pi is}-1)\Gamma(s)}I(s)\\ &= \frac{\Gamma(1-s)\sin(\pi s)}{\pi(e^{2\pi is}-1)}I(s)\\ &= \frac{e^{-\pi is}\Gamma(1-s)}{2\pi i}I(s)\end{aligned}$$

As a remark, although we only prove the formula for $\mathrm{Re}(s) \ge 1$, the integral actually converges actually for all $s \in \mathbb{C}$, and defines an entire function, so we have the equality for any s by analytic continuation. On the other hand, note the the integral over the small circle of radius δ may not converge to 0.

5. a. Since

$$\frac{z}{e^z-1} = 1 + B_1 z + B_2\frac{z^2}{2!} + B_3\frac{z^3}{3!} + \cdots,$$

   we see from residue theorem that

$$I(-n) = \int_C \frac{z^{s-1}}{e^z-1}dz = \frac{2\pi iB_{n+1}}{(n+1)!}$$

   Using the formula in the previous question, we have

$$\begin{aligned}\zeta(-n) &= \frac{e^{\pi in}\Gamma(n+1)}{2\pi i}\frac{2\pi iB_{n+1}}{(n+1)!}\\ &= (-1)^n\frac{B_{n+1}}{n+1}\end{aligned}$$

   b. We will make use of the functional equation

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s).$$

   Putting $s = 2n$, and use out the knowledge of special values of Γ and ζ (together with part a), we have

$$\begin{aligned}\zeta(2n) &= \pi^{2n-1/2}\frac{\frac{\sqrt{\pi}}{(-1/2)(-3/2)\cdots(-(2n-1)/2)}}{(n-1!)}\frac{(-1)^{2n-1}B_{2n}}{2n}\\ &= (-1)^{n+1}\pi^{2n}2^n\frac{1}{1\cdot3\cdot5\cdots(2n-1)}\frac{1}{1\cdot2\cdot3\cdots(n-1)}\frac{B_{2n}}{2n}\\ &= (-1)^{n+1}\pi^{2n}2^n\frac{1}{1\cdot3\cdot5\cdots(2n-1)}\frac{2^{n-1}}{2\cdot4\cdot6\cdots(2n-2)}\frac{B_{2n}}{2n}\\ &= (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}.\end{aligned}$$

   c. We take $\log$ derivative for the functional equation:

$$\frac{\zeta'(s)}{\zeta(s)} = \log\pi - \frac{\Gamma'((1-s)/2)}{2\Gamma((1-s)/2)} - \frac{\Gamma'(s/2)}{2\Gamma(s/2)} - \frac{\zeta'(1-s)}{\zeta(1-s)}$$

   We will take $s=1$ (or $s \to 1$), so we calculate the terms one by one. For Gamma functions, we use

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{n=1}^\infty\left(1+\frac{s}{n}\right)^{-1}e^{s/n}$$

   so

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \frac{1}{s} + \sum_{n=1}^\infty\left(\frac{1}{n} - \frac{1}{n+s}\right)$$

   We see that

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} - \gamma + O(s)$$

   and

$$\begin{aligned}\frac{\Gamma'(1/2)}{\Gamma(1/2)} &= -\gamma - 2 + \sum_{n=1}^\infty(\frac{1}{n} - \frac{2}{2n+1})\\ &= -\gamma - (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots)\\ &= -\gamma - \log 2.\end{aligned}$$

   On the other hand, we see from tutorial 6 question 1 (together with corollary 2.6 of lecture 11) that $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$, so

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \frac{-1/(s-1)^2 + O(|s-1|)}{1/(s-1) + \gamma + O(|s-1|)}\\ &= -\frac{1}{s-1} + \gamma + O(|s-1|)\end{aligned}$$

   Combining them together, we have,

$$\begin{aligned}\frac{\zeta'(1-s)}{\zeta(1-s)} &= \log\pi - \frac{1}{2}(-\frac{2}{1-s} - \gamma) - \frac{1}{2}(-\gamma - \log 2) - (-\frac{1}{s-1} + \gamma) + O(s)\\ &= \log 2\pi + O(s)\end{aligned}$$

   Finally, we get the result by noting that $\zeta(0) = B_1 = -\frac{1}{2}$ from part a.

6. Deduce the factorization from the Hadamard factorization of $\tilde{\xi}$. The constants can be obtained using 5a and 5c.
